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Abstract available at: http://ieeexplore.ieee.org/xpl/freeabs_all.jsp?arnumber=1144097

This paper appears in: Antennas and Propagation, IEEE Transactions on
Issue Date: Mar 1987
Volume: 35 Issue:3
On page(s): 353 - 355
ISSN: 0018-926X
Digital Object Identifier: 10.1109/TAP.1987.1144097
Date of Current Version: 06 January 2003
Sponsored by: IEEE Antennas and Propagation Society

A Theorem on the Moment Methods

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Abstract—**The inner product involved in the moment methods is usually an integral, which is evaluated numerically by summing the integrand at certain discrete points. In connection with this inner product, a theorem is proved, which states that the overall number of points involved in the integration must not be smaller than the number of unknowns involved in the moment method. If these two numbers are equal, a point-matching solution is obtained, irrespective of whether one has started with Galerkin's method or the least squares method. If the number of points involved in the integration is larger than the number of the unknowns, a weighted point-matching solution is obtained.**

I. Introduction

The moment methods have been widely used for solving linear operator equations in many electromagnetic problems [1]. However, sometimes certain simple facts are overlooked, which might obscure the true nature of the final solution obtained by the method used, or even lead to erroneous results or conclusions.

Manuscript received October 30, 1985; revised March 4, 1986. This work was supported in part by the Office of Naval Research under contract N00014-79-C-0598.

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IEEE Log Number 8612490.

For example, one can think, one is using the least squares technique or the Galerkin method, but the resulting solution can be identical to a point-matching solution because of a careless evaluation of the inner product. Or one might obtain unstable results due to the same negligence.

The aim of this communication is to present a simple and almost obvious theorem on the moment methods, which might be helpful in avoiding the above problems.

II. Formulation of the Theorem

Let us consider the operator equation

$$Lf = g, \tag{1}$$

where L is a linear operator (which includes certain boundary conditions), $g(X)$ is a known function (excitation), $f = f(Y)$ is an unknown function (the solution to be found), and X and Y are points in multidimensional spaces.

Equation (1) can be solved by applying the moment methods [1]. As the first step of this approach, we have to approximate the unknown function f by a finite sum:

$$f(Y) = \sum_{i=1}^{n} a_i f_i(Y), \tag{2}$$

where a_i are unknown coefficients to be determined, $f_i(Y)$ are known expansion functions which form a suitable basis, and in the limit when $n \to \infty$ must be able to represent the true solution, $f(Y)$ [2], [3]. In addition, the expansion in (2) has to satisfy certain boundary conditions as required by the original equation (1). Now we substitute (2) into (1), to obtain

$$\sum_{i=1}^{n} a_i L f_i = g. \tag{3}$$

The second step in the moment methods is to compute the coefficients $\{a_i\}$ so that the approximate equation (3) is satisfied in some sense. To that purpose we have to define an inner product of two arbitrary functions $u(X)$ and $v(X)$, $\langle u, v\rangle$, belonging to the range of the operator L. Next, we take the inner products of (3) with some suitable weighting functions $w_j(X)$, $j = 1, \cdots, n$, which form a functional basis, belonging to the range of the operator L. Thus we obtain a set of linear equations in $\{a_i\}$:

$$\sum_{i=1}^{n} a_i \langle w_j, Lf_i\rangle = \langle w_j, g\rangle, \qquad j = 1, \cdots, n, \tag{4}$$

which can be solved for $\{a_i\}$ by using either direct methods (e.g., the Gaussian elimination, or the LU transform), or by using iterative methods (e.g., the conjugate-gradient method), which are suitable for very large systems of equations.

The inner product $\langle u, v\rangle$ is usually an integral of the product of functions $u(X)$ and $v(X)$. In very few cases the inner product in (4) can be evaluated analytically, and in most practical problems it is evaluated numerically. This involves only samples of the integrand at certain points. In other words, the numerical integration formulas used to evaluate the inner product can be written in the general form as

$$\int_D p(X)\, dD = \sum_{k=1}^{m} b_k p(X_k), \tag{5}$$

where D is the domain over which the integration is performed, X is a point in that domain, b_k are weighting coefficients, and X_k are points at which the samples of the function $p(X)$ are evaluated. The function $p(X)$, in our case, equals $u(X)v(X)$.

If the same integration formula is applied to both inner products of (4), we have

$$\sum_{i=1}^{n} a_i \sum_{k=1}^{m} w_j(X_k) b_k L f_i|_{X=X_k} = \sum_{k=1}^{m} w_j(X_k) b_k g(X_k), \qquad j=1, \cdots, n. \quad (6)$$

Let us introduce the following matrices:

$$[F] = [Lf_i|_{X=X_k}]_{n\times m}, \quad (7)$$

$$[W] = [w_j(X_k)]_{n\times m}, \quad (8)$$

$$[B] = \text{diag}\,(b_1, \cdots, b_m), \quad (9)$$

$$[G] = [g(X_k)]_{m\times 1}, \quad (10)$$

$$[A] = [a_i]_{n\times 1}. \quad (11)$$

The system (6) can now be written in a compact form as

$$[W][B][F]^t[A] = [W][B][G], \quad (12)$$

where the superscript t denotes the transpose. Let us also denote

$$[V] = [W][B]. \quad (13)$$

Now we have the following equation instead of (12):

$$[V][F]^t[A] = [V][G]. \quad (14)$$

The matrix $[V]$ can be considered as a weighting matrix, which multiplies the system of linear equations

$$[F]^t[A] = [G]. \quad (15)$$

If $m > n$, the system (15) is, generally, overdetermined. Note that equations (15) are, essentially, point-matching equations, which are obtained by postulating that the approximate equation (3) is satisfied at points $X = X_k$, $k = 1, \cdots, m$. The purpose of multiplying the system (15) by $[V]$ is to obtain a system of n equations in n unknowns. The solution to (14) can be regarded as a weighted point-matching solution.

Note that it is not necessary that the same integration formula be used for each of equations (4). If different formulas are used, then, instead of (13), the elements of the matrix $[V]$ are evaluated as

$$v_{jk} = w_j(X_{jk}) b_{jk}, \; j=1, \cdots, n, \; k=1, \cdots, m, \quad (16)$$

where X_{jk} and b_{jk}, $k = 1, \cdots, m$, are the coefficients of the integration formula used for the jth equation.

Let us consider (14). In order that this equation have a unique solution for $[A]$, the matrix $[V][F]^t$ has to be regular. According to the Binet–Cauchy theorem [4], the matrix $[V][F]^t$ will be regular if the condition

$$m \geq n \quad (17)$$

is fulfilled, and if rank $[V]$ = rank $[F] = n$. If the condition (17) is violated, a unique solution does not exist (although a minimum-norm solution can be found, which might be useful in certain cases [5]).

Thus we have proved the following theorem.

If the integrals representing the inner product in a moment method solution to a linear operator equation are evaluated numerically, the overall number of points involved in the integration must not be smaller than the number of the unknown coefficients.

IV. Discussion

As the first consequence of the above theorem, let us consider the special case when $m = n$. If the inverse matrix $[V]^{-1}$ exists, both sides of (14) can be multiplied by $[V]^{-1}$, and a $n \times n$ system of linear equations is obtained. This system is, essentially, a system of point-matching equations, and, hence, the solution to (14) is identical to the point-matching solution. An example where such a situation can occur, is the following procedure. Let us adopt the expansion in (2) to be a piecewise-constant approximation (usually referred to as a pulse approximation). In that case $f_i(Y)$ is zero everywhere except over a small domain of Y, where it is constant (usually, equal to unity). If the Galerkin method is used, the weighting functions are $w_i = f_i$. The weighting functions being nonzero only over a small domain, the inner products in (4) are sometimes evaluated by using the midpoint rule, i.e., by using only one integration point per inner product. Obviously, the final result is identical to a point-matching solution, for which the matching points coincide with the integration points in the above Galerkin procedure. A similar result can be obtained if the least squares technique is used. In this case we have $w_i = (Lf_i)^*$, where the asterisk denotes the conjugate-complex value. If the pulse approximation is adopted, we have to evaluate the integrals representing the inner products over the whole domain of X where the original equation (1) is to hold, unlike the Galerkin procedure, where the integration has to be performed only over the domain where f_i is nonzero. However, the overall number of the integration points must be larger than n, unless we wish to obtain a point-matching solution (for $m = n$). Of course, this result is valid whatever weighting functions are utilized (triangular, piecewise-sinusoidal, etc.), as long as the inner products are evaluated by using numerical quadrature formulas.

The need for taking only a few integration points can arise not only in order to increase the speed of the computations, but also because of certain problems associated with the kind of the approximation adopted for the solution, which are not always clearly recognized. For example, in solving a wire-antenna problem, a piecewise-constant approximation of the current distribution can be used. (A good survey of the methods for the analysis of wire antennas is given in [6].) If the exact kernel is taken, then Lf_i has a nonintegrable singularity at the edge of the domain where f_i is nonzero. This precludes the use of both the Galerkin and the least squares technique, because the resulting inner products diverge! Yet, if the integration in evaluation of an inner product is confined to the interior of the domain where f_i is nonzero, acceptable results might be obtained, although the accuracy of the results can be much worse than with a point-matching solution (in addition to requiring a much longer CPU time), and the final result, obviously, is *not* a Galerkin (or least squares) solution (because such a solution does not exist). A singularity, though square integrable, also occurs if a piecewise-linear approximation (i.e., triangular approximation) is adopted, which can have an adverse effect especially to a least squares solution.

Another important fact which follows from (16) is that the weighting coefficients of the integration formula multiply the values

of the weighting functions w_i. In other words, the weighting functions are to a certain extent modified by the formulas for the numerical integration. The only exception is if the repeated midpoint rule is used on equally sized subdomains, in which case all the coefficients b_k are equal.

If the repeated midpoint rule is used, then the matrix $[V]$ for the Galerkin and for the least squares solution has special, simple forms. Thus, for the Galerkin solution we have $[V] = \Delta D[f_i(Y_j)]$, while for the least squares solution $[V] = \Delta D[F]^*$, where ΔD is the size of subdomains over which the functions f_i are nonzero. It is worth noting that the least squares solution is now equivalent to solving the overdetermined system (15) in the least squares sense. It is well known that with such a procedure the resulting system matrix $[F]^*[F]^t$ is positive definite and therefore the system (14) can be solved by iterative methods. However, the condition number of that matrix might become very large [7].

Finally, an analogous theorem can be formulated in connection with the moment method solution of an integral equation of the general form

$$\int_D f(Y)g(X, Y)\, dD = h(X) + \Lambda f(X), \tag{18}$$

where Λ is a parameter. Namely, if the unknown function f is approximated according to (2), then the overall number of points involved in the numerical integration must not be smaller than n.

IV. Conclusion

The inner product involved in the moment methods is usually an integral, which is evaluated numerically. Certain precautions have to be taken in evaluating the inner product in order to obtain a valid solution. In connection with this a theorem is proved, which states that the overall number of points involved in the integration must not be smaller than the number of unknowns involved in the moment method. If these two numbers are equal, a point-matching solution is obtained, rather than the desired moment method solution (e.g., a Galerkin or a least squares solution). If the number of points involved in the integration is larger than the number of the unknowns, a weighted point-matching solution is obtained. This conclusion remains valid whatever weighting functions are utilized, as long as the inner product is evaluated by using numerical quadrature formulas.

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