Transmission-Line Theory Approach to Solution of State Equations for Linear-Lumped Circuits

Djordjevic, A.R.; Reljin, B.D.; Tosic, D.V.; Sarkar, T.K.; Sch. of Electr. Eng., Belgrade Univ.

Abstract available at: http://ieeexplore.ieee.org/xpl/freeabs_all.jsp?arnumber=486163

This paper appears in: Microwave Theory and Techniques, IEEE Transactions on
Issue Date: Mar 1996
Volume: 44 Issue:3
On page(s): 479 - 482
ISSN: 0018-9480
References Cited: 5
Cited by : 2
INSPEC Accession Number: 5244928
Digital Object Identifier: 10.1109/22.486163
Date of Current Version: 06 August 2002
Sponsored by: IEEE Microwave Theory and Techniques Society

Transmission-Line Theory Approach to Solution of State Equations for Linear-Lumped Circuits

Antonije R. Djordjević, Branimir D. Reljin, Dejan V. Tošić, and Tapan K. Sarkar

***Abstract*—Linear-lumped circuits containing capacitors and/or inductors are described by differential equations. In computer-aided circuit analysis, these equations are discretized in time, thus being reduced to approximate formulas involving samples of voltages and currents. It is shown that these relations can be interpreted as exact equations for networks containing transmission lines. Hence, some features of the approximate formulas gain a clear physical interpretation. In particular, convergence and energy balance properties of the formulas become obvious, confirming advantages of the trapezoidal rule over all other formulas.**

I. Introduction

It is a common practice in microwave engineering to replace a lumped capacitor, of capacitance C, by an electrically short, open-circuited transmission-line section. If the characteristic impedance of this line is Z_c, its length l and wave propagation velocity c (assuming the line to be lossless, uniform, and distortionless), then

$$C = l/(cZ_c) = \tau/Z_c \tag{1}$$

where $\tau = l/c$ is the wave propagation time along the line. Similarly, a lumped inductor of inductance L is substituted by an electrically short, short-circuited transmission-line section, so that

$$L = \tau Z_c. \tag{2}$$

Note that the substitutions given by (1) and (2) introduce an error. For example, the impedance of a capacitor is

$$Z_C = -\mathrm{j}/(\omega C) \tag{3}$$

and the input impedance of an open-circuited transmission line is

$$Z_t = -\mathrm{j}Z_c \cot(\omega\tau). \tag{4}$$

These two functions are osculating as $\omega \to 0$, but they differ more and more as the frequency increases. A 5% difference occurs when the line length is $\lambda/8$ (where λ is the wavelength), but the capacitive character of Z_t is retained up to $\lambda/4$. As a conclusion, for a good approximation, the transmission line should be as short as possible. A similar reasoning is valid for an inductor and its short-circuited transmission-line equivalent.

On the other hand, in the lumped-circuit theory, linear capacitors and inductors can be described in the time domain by differential relations between their voltages and currents. For a given excitation, the state of the whole circuit can be described by their capacitor voltages, inductor currents, and time derivatives, which is the well-known state-equation method. In programs for computer-aided circuit analysis, these differential equations are discretized in time and reduced to formulas that involve several samples of voltages and currents [1]. Each of these formulas (further referred to as discrete formulas) is considered as an approximation to the differential equation, and the solution obtained using a formula is an approximation to the true solution of the circuit.

For a transmission line, the time-domain samples of voltages and currents can exactly be related taking the propagation delay and reflections into account [2]. The idea of this paper is to show that replacing a lumped capacitor (or an inductor) by a transmission-line section, according to (1) and (2), is equivalent to approximating the differential equation by a discrete formula, which is well known as the trapezoidal rule. Hence, this discrete formula is interpreted as an exact formula for the transmission line.

Based on this substitution, physical models for some popular discrete formulas are developed. From these models, convergence and energy balance properties of the discrete formulas become obvious.

In the paper, the analysis will be given for capacitors and the results for inductors can be deduced using the duality principle.

II. Time-Domain Transmission Line Analysis

Let us consider the transmission line approximating a lumped capacitor, characterized by (1), and relate samples of the voltage and current at one transmission-line port while the other port is open-circuited. If the z-axis is used to measure the distance along the line from the input port, the general solution of telegraphers' equations for the voltage (v) and current (i), as a function of time (t) and z, is

$$v(z, t) = f^+(t - z/c) + f^-(t + z/c) = v^+(z, t) + v^-(z, t) \tag{5}$$

$$i(z, t) = \frac{1}{Z_c} f^+(t - z/c) - \frac{1}{Z_c} f^-(t + z/c) = i^+(z, t) + i^-(z, t) \tag{6}$$

where f^+ and f^- are arbitrary functions describing the incident and reflected waves, respectively. Setting in these equations $z = 0$, we obtain

$$v(0, t) = f^+(t) + f^-(t) \tag{7}$$

$$Z_c i(0, t) = f^+(t) - f^-(t). \tag{8}$$

Adding these two equations we obtain

$$f^+(t) = \tfrac{1}{2}[v(0, t) + Z_c i(0, t)] \tag{9}$$

where we have the voltage of the wave launched from the input port. Subtracting (8) from (7) yields

$$v(0, t) = 2f^-(t) + Z_c i(0, t) \tag{10}$$

which is a Thévenin equivalent description of the transmission line, looking into the input port [2], [3]. A similar pair of equations can be derived for the other line end.

In the case considered, the incident wave, excited at the line input, travels along the line without distortion, strikes the far end after a time interval τ, gets totally reflected at the open end, and arrives back at the input port after another interval τ. Hence, for $z = 0$

$$f^-(t) = f^+(t - 2\tau). \tag{11}$$

If we uniformly sample the voltage and current at the transmission-line input at a time step $\Delta t = 2\tau$, from the above equations we have

$$v(0, n\Delta t) = v[0, (n-1)\Delta t] + Z_c i[0, (n-1)\Delta t] + Z_c i(0, n\Delta t) \tag{12}$$

Manuscript received June 11, 1995; revised November 27, 1995.

A. R. Djordjević, B. D. Reljin, and D. V. Tošić are with the School of Electrical Engineering, University of Belgrade, 11001 Belgrade, Yugoslavia.

T. K. Sarkar is with the Department of Electrical and Computer Engineering, Syracuse University, Syracuse, NY 13244-1240 USA.

Publisher Item Identifier S 0018-9480(96)01559-1.

where n is an integer. Equation (12) relates voltages and currents at the generator end at two adjacent time steps, $(n-1)\Delta t$ and $n\Delta t$. Changing the notation, instead of (12) we can write

$$v^{(n)} = v^{(n-1)} + Z_c i^{(n-1)} + Z_c i^{(n)} \tag{13}$$

where v and i denote the voltage and current at the transmission-line input, and the superscripts denote the time steps. Note that (1) can now be rewritten as

$$Z_c = \Delta t/(2C). \tag{14}$$

III. First-Order Discrete Formulas and Lumped-Element Companion Models

Consider a linear, time-invariant lumped capacitor. Its current and voltage are related by

$$i(t) = C\,\frac{\mathrm{d}v(t)}{\mathrm{d}t}. \tag{15}$$

The first-order discrete formulas relate currents and voltages at two adjacent time steps, $t^{(n)}$ and $t^{(n+1)}$. Generally, these formulas can be written in the form

$$v^{(n)} - v^{(n-1)} = [a_1 i^{(n)} + a_2 i^{(n-1)}]\Delta t/C \tag{16}$$

where a_1 and a_2 are constants such that $a_1 + a_2 = 1$ [1]. For $a_1 = 1$ and $a_2 = 0$, we obtain

$$v^{(n)} - v^{(n-1)} = i^{(n)}\Delta t/C \tag{17}$$

which is known as the backward Euler (corrector) formula. It relates the voltage and current at the present time step ($t^{(n)}$), and $v^{(n-1)}$ is known as it belongs to the previous time step. Equation (17) is the same as the relation for the voltage and current (at the present time step) for a voltage generator of electromotive force $v^{(n-1)}$ and resistance $\Delta t/C$. This generator is referred to as the companion model for (17) [1]. For $a_1 = 0$ and $a_2 = 1$, we obtain

$$v^{(n)} - v^{(n-1)} = i^{(n-1)}\Delta t/C \tag{18}$$

which is known as the forward Euler (predictor) formula. It contains only the voltage at the present time step, and $v^{(n-1)}$ and $i^{(n-1)}$ are known from the previous time step. The companion model for (18) is an ideal voltage generator, of electromotive force $v^{(n-1)} + i^{(n-1)}\Delta t/C$ (and zero resistance). Finally, for $a_1 = a_2 = 0.5$, we obtain

$$v^{(n)} - v^{(n-1)} = [i^{(n)} + i^{(n-1)}]\Delta t/(2C) \tag{19}$$

which is the trapezoidal integration rule, often used in circuit analysis [4], [5]. The companion model for (19) is a voltage generator, of electromotive force $v^{(n-1)} + i^{(n-1)}\Delta t/(2C)$, and resistance $\Delta t/(2C)$.

The above-mentioned companion models enable simplifications in the circuit analysis [1], but they do not clarify the nature of these formulas.

IV. First-Order Discrete Formulas and Transmission-Line Models

In the light of Sections II and III, we will derive here a transmission-line model for a capacitor approximately described by (16), so that (16) gives an exact description for that model. Consider an open-circuited transmission line, sketched in Fig. 1. The propagation time along the line is $\tau = \Delta t/2$ and its characteristic impedance is Z_c. A resistor of resistance R is connected in series with the line. Our objective is to find Z_c and R so to satisfy (16).

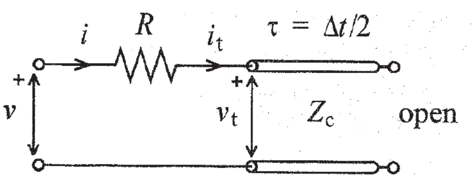

Fig. 1. Transmission-line model for a capacitor and first-order discrete formulas.

From (13), the voltage and current at the transmission-line terminals are related by

$$v_t^{(n)} = v_t^{(n-1)} + Z_c i_t^{(n)} + Z_c i_t^{(n-1)} \tag{20}$$

where the transmission-line voltage v_t is related to the voltage at the network terminals by

$$v_t = v - Ri \tag{21}$$

and $i = i_t$. Substituting v_t given by (21) for both time steps [$t^{(n)}$ and $t^{(n-1)}$] into (20), we obtain

$$v^{(n)} = v^{(n-1)} + (Z_c + R)i^{(n)} + (Z_c - R)i^{(n-1)}. \tag{22}$$

Comparing (16) and (22), we obtain $a_1\Delta t/C = Z_c + R$ and $a_2\Delta t/C = Z_c - R$. Since $a_1 + a_2 = 1$ we have further $Z_c = \Delta t/(2C)$, which is the same as (14), and $R = (a_1 - a_2)\Delta t/(2C)$. In particular, for the backward Euler formula $R = Z_c > 0$ and for the forward Euler formula $R = -Z_c < 0$. For the trapezoidal rule (and only for this formula) $R = 0$ and the model consists only of a (lossless) transmission line, the same as described in connection with (13).

These transmission-line models can easily and completely explain the stability region of the first-order formulas. This region is defined in [1] based on solving for the response of a simple serial circuit consisting of an ideal step-voltage generator, resistor (of resistance R_s), and capacitor (of capacitance C). The forward Euler formula is stable provided the time step Δt is smaller than $2R_sC$. Our transmission-line model confirms this conclusion. The total serial resistance of the circuit obtained by substituting the capacitor by the model of Fig. 1 is $R_s - Z_c = R_s - \Delta t/(2C)$, and it is positive if $R_s > \Delta t/(2C)$, i.e., if $\Delta t < 2R_sC$. If the total resistance is negative, the response of the circuit blows up.

The backward Euler formula is stable for the above example if $R_s > -\Delta t/(2C)$, while the trapezoidal formula is stable if $R_s > 0$.

However, a stable formula does not necessarily give good results, as will be demonstrated by the following example. Consider a simple resonant, lossless circuit, consisting of a capacitor [$C = (1/2\pi)$ F] and an inductor [$L = (1/2\pi)$ H]. The initial conditions for the capacitor voltage and for the inductor current are $v(0) = 0$ and $i(0) = I_0 = 1$ A, respectively. We wish to solve for the voltage of the free response, $v(t)$, $t > 0$. The exact response is sinusoidal, $v(t) = I_0\sqrt{L/C}\sin 2\pi t/T$, where $T = 2\pi\sqrt{LC}$ is the period.

If the forward Euler formula is applied to this circuit, with a time step $\Delta t = T/50$, the computed response diverges very rapidly, as shown in Fig. 2. This can easily be explained by considering the transmission-line models for the capacitor and the inductor, since the circuit contains negative resistances, and it is thus an active circuit.

If the backward Euler formula is applied, the response is "stable," but wrong, because the envelope of the voltage decays very fast, as shown in Fig. 3. Again, the transmission-line models predict such a result, because the (positive) resistances introduce losses, and the resulting quality factor of the circuit is low.

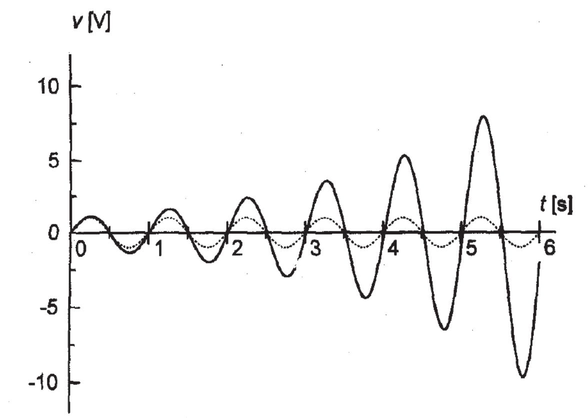

Fig. 2. Response of an LC circuit: —— obtained by the forward Euler formula, • • • • • exact.

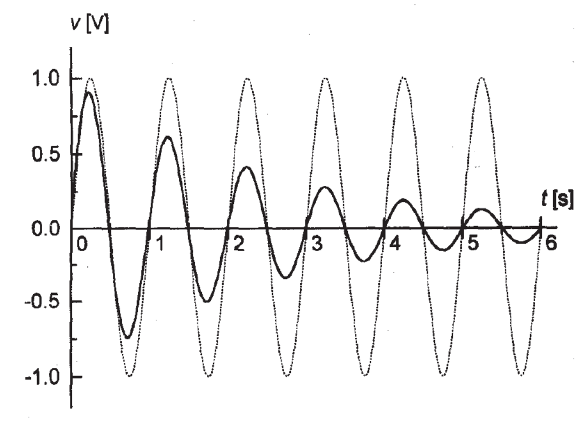

Fig. 3. Response of an LC circuit: —— obtained by the backward Euler formula, • • • • • exact.

Finally, the result obtained using the trapezoidal formula, shown in Fig. 4, has a stable amplitude, as should be expected for a lossless, passive circuit. The only visible error is in the period. This error is also predicted by our transmission-line models. Referring to (4) and a dual equation for the inductor, the period of free oscillations of the circuit formed by replacing the inductor and capacitor by their transmission-line models is given by $T_a = \pi\Delta t/\arctan(\Delta t/2\sqrt{LC})$, which is always greater than the exact period, $T = 2\pi\sqrt{LC}$. However, the difference between T_a and T decays with the time step as $(\Delta t)^2$.

V. Models for Higher-Order Formulas

Among a variety of higher-order formulas [1], transmission-line models are considered here only for backward differentiation corrector formulas (known as Gear's corrector formulas) because they have found their application in the circuit analysis [4]. When applied to a capacitor, backward differentiation corrector formulas can be put into the general form

$$i^{(n)} = -\frac{C}{\Delta t}\sum_{j=0}^{k} a_j v^{(n-j)} \tag{23}$$

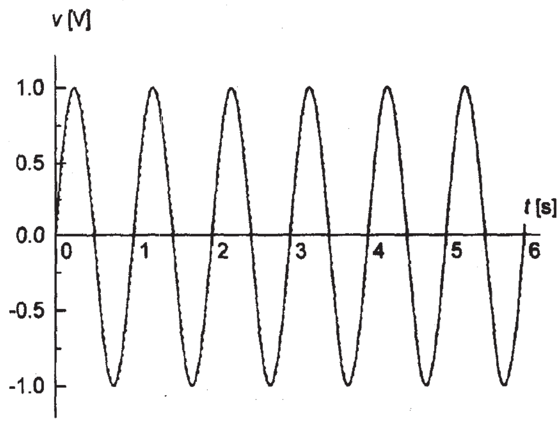

Fig. 4. Response of an LC circuit: —— obtained by the trapezoidal rule, • • • • • exact.

where k is the formula order. In addition

$$\sum_{j=0}^{k} a_j = 0 \tag{24}$$

which follows from the condition that a constant voltage must result in a zero current. Since the capacitor current in (23) is given in the form of a sum, a parallel connection of one-port networks is implied. Substituting (24) into (23) yields

$$i^{(n)} = \frac{C}{\Delta t}\sum_{j=1}^{k} a_j[v^{(n)} - v^{(n-j)}]. \tag{25}$$

Comparing (25) with (16), each network can be identified as the network of Fig. 1, with $R_j = Z_{cj}$, $Z_{cj} = \Delta t/(2Ca_j)$ and $\tau_j = j\Delta t/2$, $j = 1, \cdots, k$.

As an example, the second-order Gear's formula ($k = 2$) reads

$$i^{(n)} = [\,3v^{(n)} - 4v^{(n-1)} + v^{(n-2)}]C/(2\Delta t) \tag{26}$$

and the transmission-line model (shown in Fig. 5) consists of a parallel combination of two networks of the form shown in Fig. 1. The first network involves a positive resistance and a transmission line of a positive characteristic impedance and transit time $\Delta t/2$. The second network involves a negative resistance and a transmission line of a negative characteristic impedance and transit time Δt. It can be easily verified that the input admittance of the model of Fig. 5 has a nonnegative real part for all frequencies. Hence, formula (26) belongs to the class of "stable" formulas. When applied to a passive, lossless LC circuit, a similar behavior is observed as for the backward Euler formula. The response is damped, although with a much smaller damping factor than for the backward Euler formula.

VI. Conclusion

In computer-aided circuit analysis, differential voltage-to-current relations describing capacitors and inductors are discretized in time, thus being reduced to approximate formulas involving samples of voltages and currents. Frequently used formulas are the forward and backward Euler formulas, the trapezoidal rule, and backward differentiation corrector formulas (Gear's formulas). For these formulas, exact models were designed that contain electrically short sections of lossless transmission lines and lumped positive or negative resistors,

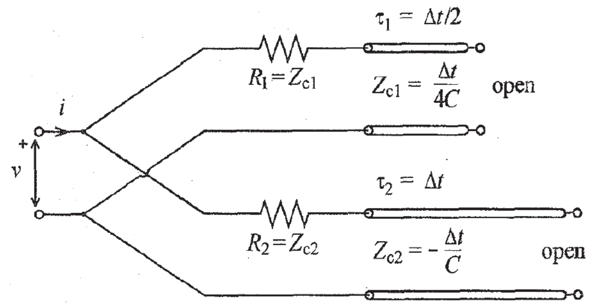

Fig. 5. Transmission-line companion model for a capacitor and second-order backward differentiation corrector formula.

except for the trapezoidal formula for which there are no resistors. These models enable a clear physical insight into convergence and energy balance properties of these discrete formulas. Transmission-line models that involve resistances imply formulas with an energy imbalance. As a result, for some formulas (e.g., the backward Euler formula or Gear's second-order formula) capacitors and inductors behave like lossy elements. This feature yields a "stable" response, but prone to large errors for low loss or lossless circuits. For some other formulas (e.g., the forward Euler formula) capacitors and inductors behave like active elements (generators), and the circuit response may easily diverge.

The only formula that has a proper energy balance is the trapezoidal rule. The corresponding model for a capacitor is an open-circuited lossless transmission-line section, while the model for an inductor is a short-circuited section and no resistors are involved. The line lengths are shortest possible for a discretized analysis, as the transit time equals one half of the time step.

These models clearly explain why the trapezoidal formula is superior to other formulas in the analysis of low loss and lossless circuits. We also note that these models are even used in microwave engineering to replace capacitors and inductors.

According to the analysis presented, we want to emphasize that the trapezoidal algorithm is the only one acceptable for the general purpose, computer-aided (numerical) circuit analysis.

References

[1] J. Vlach and K. Singhal, *Computer Methods for Circuit Analysis and Design.* New York: Van Nostrand, 1983, ch. 9 and 13.

[2] A. R. Djordjević, T. K. Sarkar, and R. F. Harrington, "Time-domain response of multiconductor transmission lines," *Proc. IEEE,* vol. PROC-75, pp. 743–764, June 1987.

[3] A. R. Djordjević, T. K. Sarkar, R. F. Harrington, and M. B. Baždar, *Time-Domain Response of Multiconductor Transmission Lines (Software and User's Manual).* Boston: Artech, 1989, section 2.3.

[4] Spice Version 2g.6, Univ. of California, Berkeley, CA, 1983.

[5] PSpice Version 5.3, MicroSim Co., Irvine, CA, 1993.

Error Bound for the Approximate Fourier Transformation Relationship for Nonuniform Transmission Lines

Roland Finkler and Rolf Unbehauen

***Abstract*—In this paper, an error bound is presented for Bolinder's well-known approximate formula [1] relating the input reflection coefficient and the local reflectivity parameter of a lossless nonuniform transmission line (NTL) via the Fourier transformation. Despite modern computers allowing an accurate analysis, Bolinder's formula is still of interest. First, it makes possible an approximate synthesis of NTL's which can be used in a subsequent optimization. Second, it supports an intuitive grasp for the electrical properties of NTL's.**

I. Exact Analysis

We consider a lossless nonuniform transmission line (NTL) with the (Laplace transforms of) voltage and current at the electrical position [2] z, $V(z, p)$ and $I(z, p)$, related by the telegrapher's equations

$$\begin{aligned}\frac{\partial}{\partial z} V(z, p) &= -pW(z)I(z, p),\\ \frac{\partial}{\partial z} I(z, p) &= -\frac{p}{W(z)} V(z, p) \quad (0 \le z \le \tau)\end{aligned} \tag{1}$$

with τ denoting the electrical length and the differentiable function $W(z)$ the characteristic impedance. Let $Z(p) = V(0, p)/I(0, p)$ be the input impedance of the NTL when terminated in the ohmic resistance R_L [Fig. 1(a)], i.e., $V(\tau, p)/I(\tau, p) \equiv R_L$. Thus

$$\Gamma(p) = \frac{Z(p) - R}{Z(p) + R} \tag{2}$$

is the input reflection coefficient with the reference resistance R.

In case of

$$\begin{aligned}R &= W(0),\\ R_L &= W(\tau)\end{aligned} \tag{3}$$

one gets [3][1]

$$\begin{aligned}\Gamma(j\omega) = \sum_{n=0}^{\infty} \int_0^{\tau} dz_1 \int_0^{z_1} dz_2 \int_{z_2}^{\tau} dz_3 \cdots \\ \int_0^{z_{2n-1}} dz_{2n} \int_{z_{2n}}^{\tau} dz_{2n+1}(-1)^n \\ \cdot P(z_1)P(z_2) \cdots P(z_{2n+1}) \\ \cdot e^{-j\omega 2(z_1 - z_2 + z_3 - \cdots - z_{2n} + z_{2n+1})}\end{aligned} \tag{4}$$

with the local reflectivity parameter

$$\begin{aligned}P(z) :&= \frac{\frac{dW(z)}{dz}}{2W(z)}\\ &= \frac{1}{2}\frac{d}{dz}\ln W(z).\end{aligned} \tag{5}$$

This result can be interpreted as follows. The reflected wave $b(j\omega)$ at the input port may be viewed as being composed of infinitesimal

Manuscript received June 29, 1995; revised November 27, 1995.

The authors are with Lehrstuhl für Allgemeine und Theoretische Elektrotechnik, Universität Erlangen-Nürnberg, D-91058 Erlangen, Germany.

Publisher Item Identifier S 0018-9480(96)01557-8.

[1] By using this result of [3], which we have checked, we make no statement about the correctness of the rest of the contents of [3].